arXiv:cond-mat/0005274v1 [cond-mat.stat-mech] 17 May 2000

Tiles and colors

Bernard Nienhuis
Instituut voor Theoretische Fysica
Valckenierstraat 65
1018 XE Amsterdam
Netherlands
e-mail nienhuis@wins.uva.nl

February 1, 2008

Abstract

Tiling models are classical statistical models in which different geometric shapes, the tiles, are packed together such that they cover space completely. In this paper we discuss a class of two-dimensional tiling models in which the tiles are rectangles and isosceles triangles. Some of these models have been solved recently by means of Bethe Ansatz. We discuss the question why only these models in a larger family are solvable, and we search for the Yang-Baxter structure behind their integrablity. In this quest we find the Bethe Ansatz solution of the problem of coloring the edges of the square lattice in four colors, such that edges of the same color never meet in the same vertex.

Keywords: Random tiling, integrable models, colorings, lattice models, quasicrystals

This paper is dedicated to Professor Rodney Baxter for his great contributions to statistical mechanics, and is particularly inspired by his lectures and seminars, where he so admirably shares with the audience his own fascinations and interests.

1 Lattices and tilings

In many models for solid state physics the lattice as a simple idealization of the crystal is a given ingredient. Of course in reality the crystal is the result of the interactions and statistics of the particles in the system. The ideal lattice can be viewed as a periodic repetition of the unit cell. This makes it periodic and it is typically also symmetric under a specific discrete rotational symmetry group. The compatibility of the translational and rotational symmetry gives the well known crystallographic restrictions on the possible rotational symmetry groups.

The quasicrystal is a state of matter which does not observe this restriction. This is possible only because it is not strictly periodic. Instead it has a property called quasiperiodicity. A quasiperiodic function in d dimensions is a restriction of a periodic function in a dimension higher than d to a d-dimensional hyperplane. The analogue for quasicrystals of the ideal lattice is the quasilattice or perfect quasiperiodic tiling. It is built as a repetition of more than one 'unit cell' also called tile. Examples of these structures are the famous Penrose tilings[1] with rhombi as tiles, or their three-dimensional analogues with Ammann rhombohedra[2, 4]. In contrast to lattices, these quasilattices may have non-crystallographic rotational symmetries, reflecting the genuinely crystallographic symmetries of the periodic functions in higher dimensions. However, these non-crystallographic symmetries are not true symmetries of the function itself, but only of the absolute value of its Fourier transform. Because it is precisely this quantity that is measured in diffraction experiments, the symmetry is nevetheless quite real.

2 Random tilings as statistical models

It was proposed by Elser[3] that these noncrystallographic symmetries can be achieved not only by a quasiperiodic arrangement, but also by an ensemble of arbitrary packings of the tiles. This happens in the same way that the high temperature phase of a statistical model reflects the symmetry of the hamiltonian, not in any particular configuration, but only in the complete ensemble and in statistical averages. For instance, in the case of the Penrose

rhombi, the angles between the edges are all multiples of $\pi/5$. Therefore if the plane is covered by copies of these rhombi, then also all the edges in the whole tiling have angles with one another which are multiples of $\pi/5$. As a consequence the continuous rotational symmetry of the plane, is reduced to at most a ten-fold discrete symmetry. The complete ensemble of tilings with these tiles will naturally have this tenfold rotational symmetry, unless the symmetry is further reduced by possible symmetry breaking schemes. This observation has led to the study of what is now known as random tilings[4, 5]. Despite their name these models have no intrinsic randomness. These are discrete statistical models of which the configurations are tilings of space by means of a limited set of tiles. They are called random only to emphasize the difference with perfect quasiperiodic tilings.

In principle one may introduce an interaction energy between adjacent tiles, but in most cases studied there is no other interaction than the full packing constraint, i.e. that the tiles cover space without holes or overlaps.

Though the phrase of random tilings came up in the study of quasicrystals, many random tilings actually live on the lattice [6]. The dimer problem[7] is a typical example, but also e.g. the hard hexagon model[8] can be viewed as a tiling with hexagons and triangles. This paper, however, is concerned with random tilings of which the tiles do not fit together on a regular lattice.

3 Solvable random tilings

On first sight the lack of an underlying lattice is a great difficulty in the analysis. However this difficulty is not essential as the configurations of the tiling can be mapped on a lattice by means of a geometric deformation[9]. With this approach Widom studied the tilings of the plane with squares and equilateral triangles[10]. Due to the fact that all angles of these objects are multiples of $\pi/6$ radians, this model exhibits twelvefold rotational symmetry. Widom showed that the transfer matrix of the tiling can be diagonalized by means of a Bethe Ansatz (BA). The corresponding Bethe-Ansatz equations were subsequently solved by Kalugin[11] in the thermodynamic limit. This solution led to an exact expression of the extensive part of the entropy of the tiling. Later De Gier and the present author have found two other cases [12, 13] of quasicrystalline random tilings which can be solved by similar techniques. The tiles are rectangles and isosceles triangles, immediate generalizations of the squares and equilateral triangles. The rotational symmetry of the maximally symmetric phase is different, namely octagonal and decagonal respectively. Some example configuration of these three solved tilings is

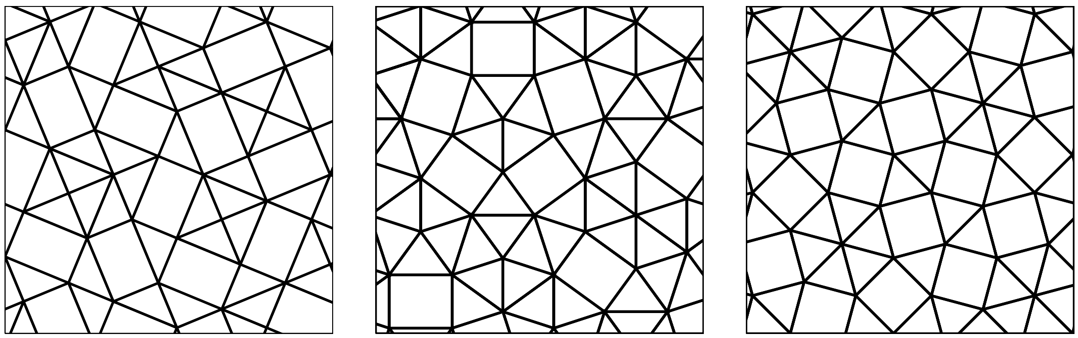

Figure 1: Some examples of configurations of the three solved rectangle-triangle tilings, the top angle of the triangle is $\pi/2$, $2\pi/5$ and $\pi/3$ respectively.

shown in Figure 1. The triangular tile has top angle $\alpha = 2\pi/n$, where the integer n takes the values 4, 5 and 6. The rectangle is simply defined to have sides matching in length with the legs (length 1) and the base (length $2\sin\alpha/2$) of the triangle.

An obvious question at this point is in what way these tilings are different from other rectangle-triangle tilings, in which the angle α is another rational or even irrational fraction of 2π. To see this it is necessary to consider in more detail the geometry of the tiling. Consider a vertex of the tiling, where i triangles meet with their top angle α, j triangles with their base angle, and k rectangles. Then obviously

$$i\alpha + j\frac{\pi - \alpha}{2} + k\frac{\pi}{2} = 2\pi, \tag{1}$$

and the sum $j + k$ is even. Irrespective of the value of α three solutions of these conditions always exist, namely $(i, j, k) = (0, 0, 4)$, $(1, 2, 2)$ and $(2, 4, 0)$. Tilings with only vertices of these types we will call generic tilings. In all configurations of these tilings it is possible to vary continuously the angle α, while the base angle and the length of the edges vary accordingly. Therefore the angle α plays no role in counting the number of ways the plane can be tiled. However, the restriction to this limited set of vertex types is so severe that these tilings have zero entropy: any finite region can be tiled in at most one way. For this reason these models are not of great interest from the view point of statistical physics.

The tilings are possibly more interesting when α is chosen such that other combinations for (i, j, k) are possible. Though there are many ways to allow for other solutions of (1), there are other conditions to cope with. The total

number of base angles of an entire configuration is always twice the number of top angles. Therefore when there is a vertex in which $2i < j$ there must in the same configuration also be a vertex with $2i > j$. Therefore the only values of α giving rise to other than generic tiling configurations, are those which admit solutions of (1) both with $2i < j$ and $2i > j$. The task to find these values of α is elementary, but tedious. The result is that $\alpha = 2\pi/n$ with $n = 3$, 4, 5 or 6. All other values of α give only generic tilings. The cases $n = 4$, 5 and 6 are mentioned above and are precisely those tilings that have been solved by means of the Bethe Ansatz[11, 12, 13]. The case $n = 3$ has not been discussed in the literature before. The tiles, a triangle with sides 1, 1 and $\sqrt{3}$, and a rectangle with sides 1 and $\sqrt{3}$ are such that the vertices of this tiling together with the mid-points of the rectangles form precisely the sites of a triangular lattice. It is tempting to believe that this tiling is also solvable by BA, but this does not appear to be the case.

It may still be true that the rectangle-triangle tilings with $n = 4$, 5 and 6 are members of an infinite sequence of solvable models. It may be that for higher values of n the two tiles, the triangle and the rectangle do not suffice, and one may need to introduce more tiles as n increases. On the same token it may be that for $n = 3$ the two tiles are already too many, and one should work only with the triangle. This indeed gives a solvable tiling with finite entropy: when there are no rectangles pairs of triangles always share their long side, and thus form a rhombus. This rhombus tiling has been studied in many guises[14].

4 Integrability of the square-triangle tiling

In almost all cases where the BA approach to the diagonalization of a transfer matrix or quantum Hamiltonian is effective, these operators are members of a commuting family[15] The commutativity of this family is proven by the fact that the local Boltzmann weights satisfy the Yang-Baxter (YB) equation[15]. Such a connection to a YB structure is not apparent for the solvable random tilings. However, in the case of the square-triangle tiling, a connection has been found[16]. It turns out that the square-triangle model is equivalent to a limit of a known vertex model (associated with the affine Lie algebra $\mathrm{A}_2^{(1)}$), provided with fields of the Perk-Schultz[17] type. This vertex model does solve the YB equation. Without repeating the complete argument I will summarize the connection, and make some comments for later reference. For the derivation the reader is referred to[16].

The Boltzmann weights denoted as $W(\alpha, \beta; \gamma, \delta)$, with as successive ar-

guments the states of the (left, bottom; top, right) legs of the vertex.

$$\begin{aligned} W(\mu,\mu;\mu,\mu) &= X_\mu^2 \sinh(u+\lambda) \\ W(\mu,\nu;\mu,\nu) &= X_\mu X_\nu \mathrm{e}^{u\,\mathrm{sgn}(\nu-\mu)} \sinh\lambda \\ W(\mu,\nu;\nu,\mu) &= X_\mu^2 (Y_{6-\mu-\nu})^{2\,\mathrm{sgn}(\mu-\nu)} \sinh u \end{aligned} \tag{2}$$

where μ and $\nu \neq \mu$ take the values 1, 2 and 3. The limit involves both the spectral parameter u and the field parameters X_μ and Y_μ. The square-triangle model is recovered when the limit is taken in two steps. In the first step the spectral parameter is taken to $u = -\lambda$, at which some of the vertex weights vanish. This is a point of special symmetry where the vertices of the square lattice can be factorized in vertices of the hexagonal lattice[18]. In fact both the spatial symmetry of the hexagonal lattice and the internal permutation symmetry of the three vertex states are fully observed. At this point the model is equivalent to the three-coloring problem of the edges of the honeycomb lattice such that equally colored edges never meet in a vertex. This problem was solved by Baxter[19]. In the second step a combination of the Perk-Schultz fields is taken to an extreme limit. $X_1^{-1} = X_2 = X_3 = Y_1^{-1} = Y_2^{-1} = Y_3^{-1} = x$ and take $x \to 0$. In this limit a few of the remaining Boltzmann weights vanish as well. Because in the first limit the spectral parameter as a free variable is lost, so is also the YB structure. In order to retain the spectral parameter one might wish to take the field limit first. However in that procedure one of the vertex weights would diverge, or after suitable normalization, be the only one to survive. Thus we see the two limits do not commute. As a consequence we come to the counter-intuitive conclusion that the Boltzmann weights of the square-triangle tiling are a limit of a solution to the YB equation, but these weights themselves are not a proper solution. It thus appears that the solvability of square-triangle model is related to the YB equation, albeit somewhat remotely.

In the remainder of this paper we investigate in more detail the solvable rectangle-triangle tiling with octagonal symmetry, and in particular discuss the question if it also can be viewed as a limit of a YB-solvable model. For this purpose we review the solution, though in an approach different from that published before[12]. Here we try to keep the symmetry of the tiling as much as possible.

5 The octagonal rectangle-triangle tiling

This model was first considered by Cockayne[20] in a slightly different language. The plane (or a finite periodic section of it) is tiled with rectangular,

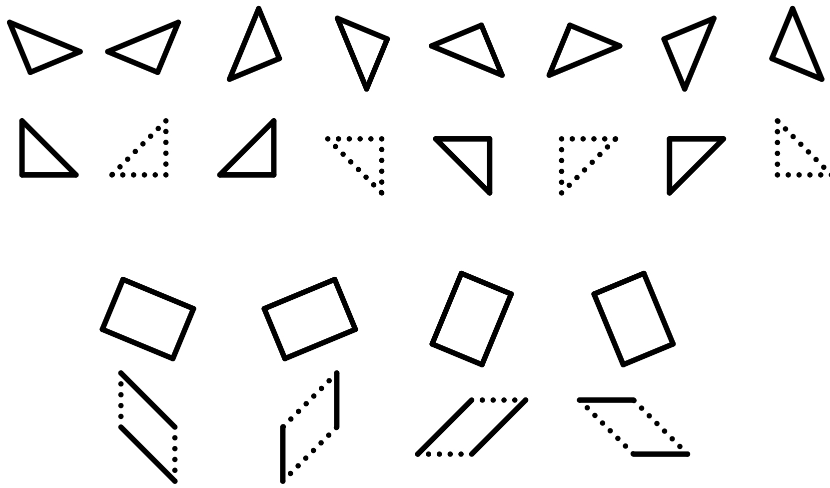

Figure 2: Deformations of the triangular and rectangular tiles in their respective orientions.

isosceles triangles and rectangles of which the sides, length 1 and $\sqrt{2}$, match those of the triangle. One may, as in a canonical ensemble fix the relative density of each of the tiles, or as in the grand canonical ensemble control the relative density by means of an activity variable. The tiles are not permitted to overlap, and may leave no space uncovered. Since all the angles in the tiles are multiples of $\pi/4$, the angle between any two edges in a tiling configuration must also be such multiple. Therefore the rectangles may occur in four, and the triangles in eight different orientations, shown in figure 2.

As a first step in the analysis the tiles are deformed, such that they are commensurable with a regular square lattice but continue to cover the plane without holes or overlap. The deformation is illustrated in figure 2 and may be described as follows. Let all the edges in the original tiling make an angle with the horizontal equal to an odd multiple of $\pi/8$ radians. The short edges are rotated over $\pi/8$ to the left or to the right, such that they end up horizontally or vertically. The long edges are also rotated over $\pm\pi/8$ but such that they end up in one of the diagonal directions. As a result the triangles are rotated rigidly and the rectangles are deformed into parallelograms. The deformed tiles fit precisely on the square lattice, and any tiling configuration can be viewed as the state of a lattice model. The tile edges in the resulting lattice are marked by either solid or dotted lines to distinguish the original orientations. This is necessary because two differently oriented edges are mapped onto the same orientation. The fact that the mapping, while it deforms the entire configuration, does not create any holes or overlaps, follows

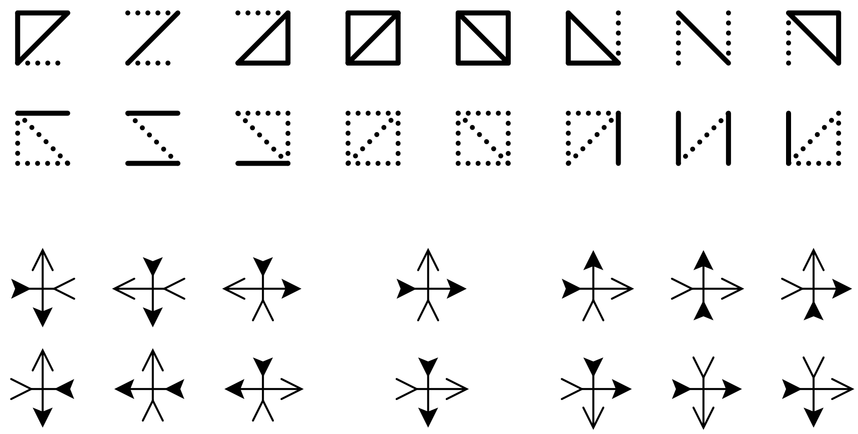

Figure 3: The top two lines show the possible configurations of an elementary square of the lattice model corresponding to the octagonal tiling model. The bottom two lines show the corresponding configurations of the fourteen-vertex model.

directly from the observation that it is defined as a mapping of the edges, i.e. the shared boundary between adjacent tiles.

The configurations of the original tiling are mapped one-to-one on configuratings of the lattice tiling. Therefore the combinatorial problem of counting the number of possible tilings seems unaltered. However, because the overall shape and the area of a tiling is altered by the mapping, the problem of tiling a given section of the plane with a given number of tiles is not the same. In the thermodynamic limit this distinction will be insignificant. The change of area can be accounted for, and the shape does not matter for the thermodynamic functions.

The lattice representation of the tiling can be compactly described by listing the possible states of each elementary square of the lattice, as shown in figure 3. The sides of the elementary squares can be in one of four states, and represent either a short edge of the tiling, or a diagonal of a rectangle. The local configurations of the elementary faces of the lattice can be represented as fourteen different states of a vertex model which has four possible states for each edge. Figure 3 shows a representation in terms of two types of arrows, which will be called open and filled. This notation makes immediately evident two conservation laws, the net flux of open and that of filled arrows. We will refer to this lattice model as fourteen-vertex model.

6 Bethe Ansatz equations

A coordinate BA formalism can be set to diagonalize the vertex model shown in figure 3. For the reference state we choose the open arrow up. The quasi-excitations are the locations with a vertical filled arrow, up or down. The open arrow down serves as a bound state between these two different excitations. Besides the conservation of open and filled flux, there is an additional conservation law, which is less evident. It turns out that at every application of the transfer matrix the excitations move one step either to the left or to the right. Therefore the lattice can be divided into two sublattices such that the number of excitations on each sublattice remains constant from row to row. As a result there are in total three conserved number and correspondingly three families of BA variables.

In order to find the most general solution the vertex weights of the model are kept general. As the BA eigenstate takes form, however, several restrictions on the weights are necessary. In the first place it turns out that the weights of the vertices almost factorize into weights associated with the tiles. Complete factorization would imply completely non-interacting tiles (apart from the ban on holes and overlaps). The one exception is that where two triangles form a square by sharing their long edge, this configurations has an extra weight 1/2. The tiles may have different weights in the different orientation, but there are restrictions on these weights. We denote the weights of the oriented triangles as t_j, with $j = 1, \cdots, 8$ and of the rectangles as r_j, with $j = 1, \cdots, 4$, both in the order in which they are shown in figure 2. Then the restrictions are:

$$\begin{aligned} t_1 t_5 &= t_3 t_7 \\ t_2 t_6 &= t_4 t_8 \\ r_1 r_3 &= \pm r_2 r_4 \end{aligned} \tag{3}$$

Because we favor a statistical interpretation of the tiling we choose the positive sign in the last line.

The resulting BA equations are, for a lattice with even horizontal size L, and periodic boundary conditions.

$$\begin{aligned} x_p^L &= (-)^{n_x+1} A \prod_{k=1}^{n_\lambda} \left(\lambda_k - x_p^2\right) \\ y_q^L &= (-)^{n_y+1} A \prod_{k=1}^{n_\lambda} \left(\lambda_k - y_q^2\right) \\ 1 &= (-)^{n_\lambda+1} B \prod_{p=1}^{n_x} \left(\lambda_k - x_p^2\right) \prod_{q=1}^{n_y} \left(\lambda_k - y_q^2\right) \end{aligned} \tag{4}$$

In terms of the solutions x_p, y_q and λ_k of these equations the eigenvalue of the transfer matrix reads

$$\Lambda = C \prod_{k=1}^{n_\lambda} \lambda_k \prod_{p=1}^{n_x} x_p^{-1} \prod_{q=1}^{n_y} y_q^{-1} \tag{5}$$

The constants are expressed in the tile weights:

$$\begin{aligned} A &= \left(\frac{t_1 t_5 r_3}{t_2 t_6 r_4}\right)^{n_\lambda} \\ B &= \left(\frac{t_1 t_5}{r_4}\right)^{L} \left(\frac{t_1 t_5 r_3}{t_2 t_6 r_4}\right)^{n_x + n_y} \\ C &= (t_1 t_5)^L \left(\frac{r_1}{t_1 t_5}\right)^{n_x + n_y - n_\lambda} \left(\frac{r_2}{r_1}\right)^{n_\lambda} . \end{aligned} \tag{6}$$

The form of these BA equations is typical for tiling models. The eigenvalue expression is a simple product rather than a sum of, say four, products, according to the number of edge states. The factors in the BA equations are binomial rather than rational or trigonometric.

7 Search for a Yang-Baxter structure

Even though the tiling model has a number of free parameters in the solution, none of those plays the role of a spectral parameter. The only way that the parameters feature in the eigenvalue is via the combination C as an overall factor. As the BA equations are the most general given the set of vertex configurations with non-zero weight, the only way to introduce a spectral parameter is to include other vertex configurations. The natural choice is to include only configurations that satisfy the flux conservation of both types of arrows. Our attempts in this direction have all failed, as they led only to null solutions. Incapable of an exhaustive search, we can not make strong statements about this possibility.

However, the situation may be similar to the case of square-triangle model, described above. If that is so it would be futile to seek solutions of the YB equations which include the fourteen-vertex model. Instead this model would be a singular limit of such a solution, and not a true member. It is on this assumption that we proceed. Guided by the results concerning the integrability of the square-triangle model, we seek a vertex model of which the vertex configurations include those of the fourteen-vertex model, but have a higher symmetry. This would still have a fixed value of the spectral parameter, and permit the introduction of field parameters, such that in a special limit the fourteen-vertex model is recovered.

A natural choice of a symmetrized version of the fourteen-vertex model is extending the permitted vertex states with those obtained by arrow inversion from the original fourteen. The resulting set of vertices are twenty-four states in which one open and one filled arrow enter and leave the vertex. We propose that it is possible to assign weights to these twenty-four vertex states such that (i) they form a member of a solution of the YB equation, and (ii) permit a limit in which they reduce to the fourteen vertices of the octagonal tiling problem. To be more precise we propose that there exists a solution of the YB equation

$$\sum_{\mu'',\nu'',\gamma} W''(\mu,\nu;\nu'',\mu'')W(\mu'',\beta;\gamma,\mu')W'(\nu'',\gamma;\alpha,\nu') = \tag{7}$$

$$\sum_{\mu'',\nu'',\gamma} W''(\mu'',\nu'';\nu',\mu')W'(\nu,\beta;\gamma,\nu'')W(\mu,\gamma;\alpha,\mu''), \tag{8}$$

such that the Greek symbols take the four arrow states as values. Part (i) of the proposal states that the symbol W is non-zero only for the twenty-four cases of its arguments when one open and one filled arrow enter and leave the corresponding vertex. Guided by the YB structure of the square-triangle model, we expect that the symbols W' and W'' may have nonzero entries besides these ones. Part (ii) of the proposal states that the solution manifold permits a limit to be taken in which the symbol W is reduced to weights of the fourteen vertices corresponding to the octagonal tiling model (figure 3). In this limit some elements of the symbols W' or W'' may well diverge.

Before discussing tests of this proposal we make a few remarks concerning the symmetry of the model. Consider a configuration of these vertices on a square lattice. The edge states are denoted by the kind of arrow, open or filled, and by the direction, up or down for vertical edges and left or right for horizontal ones. If the sites of the lattice are divided into two square sublattices A and B, the direction of the arrows can be denoted as A to B or vice versa. With this labelling, the four edges incident in the same vertex are always in different states. In other words the configurations of the 24-vertex model are in one-to-one correspondence to the colorings of the edges of the square lattice with four colors, such that in no vertex two edges of the same color meet. Clearly in this coloring problem the four colors can be freely permuted without altering the ensemble of coloring configurations. Since any of the twenty-four vertex configurations can be turned into any other one by a suitable permutation of the four edge states, the only weight assignment invariant for these permutations gives all vertices equal weight. It is likely that if there is an integrable manifold in this 24-vertex model, it includes the symmetric point where all weights are equal, equivalent to the four-coloring problem of the edges of the square lattice.

In order to test the above proposal we set up and attempt to solve the YB equations (8). The most restrictive and simple approach is to insert for all three symbols, W, W' and W'' just the twenty-four weights described above, setting all other weights equal to zero. This turns out to lead only to a trivial null solution. Thus, the full solution, if it exists reuires more than the twenty-four vertices. A natural extension is with those vertices in which the total flux of each types of arrows is still conserved. Those vertex configuration are the well-known six-vertex configuration, now in two types, those with only open arrows and those with only filled arrows, twelve configurations in total. Together with the original twenty-four vertices this makes thirty-six. We have not succeeded in solving the YB equations for these thirty-six weights for each of the symbols W, W' and W'', due to the complexity of the problem. However, by allowing for W only the original twenty-four weights while retaining the full thirty-six weights for W' and W'', the the problem is considerably simpler and could in fact be solved. Unfortunately, the resulting weights for W can not be made all positive. Therefore, contrary to our expectation the four-coloring problem is not included in the solution. Even so, part (i) of the proposal is verified, albeit with some negative weights. We note that the twenty-four weights can be made equal in absolute value, even though they can not all be made positive.

The solution of this reduced set of YB equation has still a great number of free parameters. This freedom permits the reduction of the weights W to only the fourteen non-zero weights of the octagonal tiling model. Interestingly, in that limit the same restriction on the vertex weights is found as those following from the BA, with the sign in the third line of (3) no longer free, but negative. In other words, one, or three of the weights of the oriented rectangles must be negative. Thus we have not found the YB structure behind the integrability of the octagonal tiling problem, but something intriguingly close to it. We can not recover the tiling problem from a YB structure, but then the tile weights can not be all positive. Thus part (ii) of the proposal is almost verified: the solution to the YB equation has a limit which up to signs reproduces the weights of the tiling model.

8 The four-coloring problem

In the previous section we encountered the combinatorial problem of coloring the edges of the square lattice in four colors, such that two equally colored edges never meet in the same vertex. This four-coloring problem has been studied also in its own right[21]. It is related to fully packed loop models on the square lattice[22] which has application to physics of polymers in the

melt[24]. It is also related to the Hamiltonian walk problem [23].

As described above, we have not succeeded in finding a solution to the YB equation which includes this model, at least not the coloring problem in which all configurations have positive weight. This does not imply that the model is not integrable. To yet have an indication of its integrability we have attempted to construct BA eigenectors to the transfer matrix. We used the formulation of the 24-vertex model. Thus the edges of the square lattice wrapped on a cylinder of even circumference L all carry an open or filled arrow, pointing along the edge. The configurations in which one arrow of each type points into each vertex have weight one, all other configurations have weight zero.

The coordinate (nested) BA approach indeed yields eigenvectors to the transfer matrix, rather similar in structure to those of the octagonal tiling model (4). With respect to a reference state, all open arrow up, the excitations are the filled arrows. The open arrow down serves again as a bound state of two opposite filled arrows. With every application of the transfer matrix the excitations that do not form a bound state, move a one step, to the right or to the left. As a consequence not only the number of excitations is conserved, but also their distribution over two sublattices.

Precisely as in the fourteen-vertex model, two families of variables give the momenta of the excitations on the even and odd sublattice respectively, and a third set of variables is associated with the distribution of the filled down arrows among all of the filled arrows.

The resulting BA equations read in suitable variables

$$
\begin{aligned}
\left(\frac{1+u_j}{1-u_j}\right)^{L/2} &= -\left(\prod_{m=1}^{n_w}\frac{w_m-u_j+1}{u_j-w_m+1}\right)\left(\prod_{k=1}^{n_u}\frac{u_j-u_k+2}{u_j-u_k-2}\right)\\
\left(\frac{1+v_j}{1-v_j}\right)^{L/2} &= -\left(\prod_{m=1}^{n_w}\frac{w_m-v_j+1}{v_j-w_m+1}\right)\left(\prod_{k=1}^{n_v}\frac{v_j-v_k+2}{v_j-v_k-2}\right)\\
-1 &= \left(\prod_{j=1}^{n_u}\frac{w_m-u_j+1}{u_j-w_m+1}\right)\left(\prod_{k=1}^{n_v}\frac{w_m-v_k+1}{v_k-w_m+1}\right)\left(\prod_{l=1}^{n_w}\frac{w_l-w_m+2}{w_l-w_m-2}\right)
\end{aligned}
\tag{9}
$$

The eigenvalue can be written in terms of the solutions of these equations

$$
\begin{aligned}
\Lambda &= \left(\prod_{j=1}^{n_u}\frac{1+u_j}{1-u_j}\right)^{1/2}\left(\prod_{k=1}^{n_v}\frac{1+v_k}{1-v_k}\right)^{1/2}\left(\prod_{m=1}^{n_w}\frac{2-w_m}{w_m}\right)\\
&+ \left(\prod_{j=1}^{n_u}\frac{1-u_j}{1+u_j}\right)^{1/2}\left(\prod_{k=1}^{n_v}\frac{1-v_k}{1+v_k}\right)^{1/2}\left(\prod_{m=1}^{n_w}\frac{-2-w_m}{w_m}\right)
\end{aligned}
\tag{10}
$$

These equations have been derived for the sectors where the excitations are relatively sparse. We have no proofs for these equations in full generality. We have numerically verified them for arbitrary sectors of lattices up to twelve sites in the horizontal direction. When the roots of the equation include a $w_m = 0$ the eigenvalue is undetermined from these expressions. This ambiguity could be resolved if we could find the eigenvalue expression with a spectral parameter. Altogether the BA approach shows that indeed this four-coloring problem is integrable.

These BA equations look very much like those derived by Martins[25] for mixed SU(N) vertex models for the case $N = 4$. However, the sign of some of the factors is different (those involving both u and w or both v and w). The eigenvalue (10), aside from the absence of a spectral parameter here, differs from also from that of Martins in an overall factor, which depends on the BA roots. In fact the comparison inspires to introduce a spectral parameter u in our expression for the eigenvalue. This process may seem arbitrary, but it is highly constrained for the following consideration. It is known[15] that the BA equations follow directly from the expression for the eigenvalue by the requirement that the eigenvalue be an entire function of the spectral parameter. Here we follow the reverse argument, we know the BA equations and the eigenvalue for one value of the spectral parameter, and propose a specific dependence on the spectral parameter such that the poles from the different terms in the expression cancel against each other as a consequence of the BA equations.

$$\begin{aligned}
\Lambda(u) = &\left(\prod_{j=1}^{n_u} \frac{1+u_j}{1-u_j}\right)^{1/2} \left(\prod_{j=1}^{n_v} \frac{1-v_j}{1+v_j}\right)^{1/2} \times \\
&\left[(1-u^2)^{L/2} \left(\prod_{j=1}^{n_v} \frac{1+v_j-2u}{1-v_j+2u}\right) \left(\prod_{j=1}^{n_w} \frac{2-w_j+2u}{w_j-2u}\right)\right. \\
&+(1-u^2)^{L/2} \left(\prod_{j=1}^{n_u} \frac{1-u_j+2u}{1+u_j-2u}\right) \left(\prod_{j=1}^{n_w} \frac{-2-w_j+2u}{w_j-2u}\right) \\
&+(u+u^2)^{L/2} \left(\prod_{j=1}^{n_u} \frac{-3-u_j+2u}{1+u_j-2u}\right) \\
&\left.+(u^2-u)^{L/2} \left(\prod_{j=1}^{n_v} \frac{-3+v_j-2u}{1-v_j+2u}\right)\right]
\end{aligned} \tag{11}$$

The value $u = 0$ corresponds to the original expression (10). One check of this proposal is that it now allows us to calculate the eigenvalue unambiguously

also in those cases where one of the w_m roots vanishes. Indeed it turns out that these cases the limit $u \to 0$ of the expression correspond to eigenvalues of the transfer matrix that were undetermied before.

The eigenvalue expression (11) together with the BA eigenvectors of the transfer matrix of the 24-vertex model unambiguously defines a matrix. If this matrix can be built up from local Boltzmann weights is yet to be verified. The whole expression has some unusual features, such as the obvious overall factor, independent of the spectral parameter, and involving a square root.

9 Summary

Quasicrystals show rotational symmetries which are forbidden by the rules of crystallography. This is possible because quasicrystals are aperiodic. From the fact that quasicrystals exist we must conclude their structure is a minimum of the free energy. It can be argued that in quasicrystals, more than in crystals, the entropy is, compared with the energy, plays a significant role in determining this minimum. For these materials random tilings serve as model systems.

This paper reviews some results of two-dimensional random tilings of which the entropy has been calculated exactly. These models have phases with octagonal, decagonal and dodecagonal spatial rotational symmetry. They appear as three member of an infinite sequence, but there properties, among which solvability, uniquely set them apart from the rest of the sequence. We have discussed the question why this is the case.

While these models have been solved by means of the Bethe Ansatz, an underlying Yang-Baxter structure is not apparent. Only in the dodecagonal case the relation with a solution to the YB equation has been known for a few years. Here the connection involves a model family which includes the coloring problem of the edges of the hexagonal lattice with three colors, such that the edges meeting in a vertex have different colors. The tiling model is a singular limit of this family.

This paper takes a few steps at finding a similar connection for the solved tiling model with octagonal symmetry. Its transfer matrix also permits a BA solution. The search for the YB structure behind this integrability is not completed. However, the trace seems to lead, just as in the dodecagonal case, by a coloring problem. In this case it is the problem of coloring the edges of the square lattice in four colors such that nowhere two edges with the same color meet in a vertex. We give the BA equations of this problem, thus showing that the model is integrable.

Acknowledgments

The author wishes to thank the organizers of "Baxter's revolution in mathematical physics", and R. Tateo, S.O. Warnaar, J.C. de Gier and D. dei Cont for many valuable discussions.

References

[1] R. Penrose, The role of aesthetics in pure and applied mathematical research, *Bull. Inst. Math. Appl.* **10** (1974), 266.

[2] L.J. Shaw, V. Elser, and C.L. Henley, Long-range order in a three-dimensional random-tiling quasicrystal *Phys. Rev.* B **43** (1991), 3423

[3] V. Elser, Comment on "Quasicrystals: a new class of ordered structures". *Phys. Rev. Lett.* **54** (1985), 1730, and Indexing problems in quasicrystals diffraction. *Phys. Rev.* B **32** 1985), 4892.

[4] L.H. Tang, Random-tiling quasi-crystal in 3 dimensions, *Phys. Rev. Lett.* **64** (1990), 2390

[5] K.J. Strandburg, Random-tiling quasicrystal, *Phys. Rev.* B **40** (1989), 6071

[6] Richard C, Hoffe M, Hermisson J, M. Baake, Random tilings: concepts and examples, *J. Phys.* A **31** (1998), 6385

[7] P. W. Kasteleyn, The statistics of dimers on a lattice, *Physica* **27** (1961), 1209, and H. N. V. Temperley and M. E. Fisher, *Phil. Mag.* [8th Ser.] **6** (1961), 1061.

[8] R.J. Baxter, Hard hexagons: exact solution, *J. Phys.* A **13** (1980), L61

[9] C.L. Henley, Random tilings with quasicrystal order: transfer-matrix approach, *J. Phys.* A **21** (1988), 1649.

[10] M. Widom, Bethe Ansatz solution of the square-triangle random tiling model. *Phys. Rev. Lett.* **70** (1993), 2094.

[11] P.A. Kalugin. The square-triangle random-tiling model in the thermodynamic limit. *J. Phys.* A **27** (1994), 3599.

[12] J. de Gier and B. Nienhuis, Exact solution of an octagonal tiling model, *Phys. Rev. Lett.* **76** (1996), 2918-2921 and J. de Gier and B. Nienhuis. The exact solution of an octagonal rectangle-triangle random tiling. *J. Stat. Phys.* **87**, 415.

[13] J. de Gier and B. Nienhuis. Bethe Ansatz solution of a decagonal rectangle-triangle random tiling, *J. Phys.* A **31** (1998), 2141-2154.

[14] H.W.J. Blöte and H.J. Hilhorst, Roughening transitions and the zero-temperature triangular Ising antiferromagnet, *J. Phys.* A **15** (1982), L631

[15] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics*, Academic Press, London (1982).

[16] J. de Gier and B. Nienhuis, Integrability of the square-triangle random tiling model. *Phys. Rev.* E **55** (1997), 3926.

[17] J.H.H. Perk and C.L. Schultz, New families of commuting transfer matrices in q-state vertex models, *Phys. Lett.* **84A** (1981), 407

[18] N.Y. Reshetikhin, A new exactly solvable case of an O(n) model on a hexagonal lattice *J. Phys.* A **24** (1991), 2387

[19] R.J. Baxter, Coloring of a hexagonal lattice, *J. Math. Phys.* **11** (1970), 784

[20] E. Cockayne, Atomistic octagonal random-tiling model, *J. Phys.* A: Math. Gen. **27** (1994), 6107

[21] J. Kondev and C.L. Henley, Four-coloring model on the square lattice - a critical ground-state, *Phys. Rev.* B **52** (1995), 6628

[22] J.L. Jacobsen, On the universality of fully packed loop models, *J. Phys.* A **32** (1999), 5445and S. Higuchi, Compact polymers on decorated square lattices *J.Phys.* A **32** (1999), 3697

[23] M.T. Batchelor, H.W.J. Blöte, B. Nienhuis and C.M. Yung, Critical behavior of the Fully Packed Loop model on the square lattice, *J. Phys.* A **29** (1996), L399

[24] J.L. Jacobsen and J. Kondev, Field theory of compact polymers on the square lattice, *Nucl. Phys.* B **532** (1998), 635, and J. Kondev and J.L. Jacobsen, Conformational entropy of compact polymers, *Phys. Rev. Lett.* **81** (1998), 2922

[25] M.J. Martins, Integrable mixed vertex models from braid monoid algebra, (solv-int/9903006) in *Statistical physics on the eve of the 21-st century*, eds. M.T. Batchelor, L.T. Wille, Vol 14 of *Series on anvances in Statistcal Mechanics*, World Scientific, Singapore 1999.